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FILTERING WITH TWO SIDED FILTRATIONS

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Abstract The value of a diffusion at an intermediate point is observed through noisy observations on each side. Corresponding semimartingale decompositions and recursive filtering equations are obtained.

1. INTRODUCTION

Suppose a signal process x_s is observed through a noisy observation process y_s for $s \in [0,1]$. A situation is considered where the observations y_s are known both for $0 \leq s \leq t < 1/2$, giving a σ-field Y_t, and for $1/2 < 1-t \leq s \leq 1$, giving a σ-field Z_{1-t}, and we wish to estimate $x_{1/2}$ say, or some function $F(x_{1/2})$ of $x_{1/2}$. In mean square the best estimate is

$$E[F(x_{1/2}) \mid Y_t \vee Z_{1-t}]$$

and a recursive form of this estimate is obtained. Such problems possibly arise in reconstructing images from noisy data if one wishes to estimate the signal at linear location $1/2$ based on observations from 0 to t and $1-t$ to 1 either side. This estimation involves the two σ-fields, Y_t, increasing in the positive t direction, and Z_{1-t} increasing in the negative t direction. When x and y are diffusions the recursive equation for

$$E[F(x_{1/2}) \mid Y_t \vee Z_{1-t}]$$

is derived below. Detailed calculations can be found in [4].

The construction is related to the decomposition of diffusions with respect to enlarged filtrations, and to Brownian bridges. To

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illustrate the methods the decomposition of a Brownian motion with respect to a two-sided filtration is first obtained.

2. BILATERAL BRIDGES

The technique below was first used by Ito [5] to discuss the reverse time decomposition of a Brownian motion.

Convention 2.1. *We shall assume all filtrations are complete and right-continuous.*

Suppose $\{B_t\}$, $0 \le t \le 1$ is a standard Brownian motion on (Ω, F, P). Write

$$F_t = \sigma\{B_s : 0 \le s \le t\}$$

$$G_t = \sigma\{B_s : t \le s \le 1\}$$

and, (see Convention 2.1), consider the forward and backward filtrations $\{F_t\}$, $\{G_t\}$, $0 \le t \le 1$. For $0 \le t \le \frac{1}{2}$ consider the two-sided filtration $\{F_t \vee G_{1-t}\} = \{H_t\}$.

Lemma 2.2. For $0 \le t < \frac{1}{2}$, *B is a $\{H_t\}$ semimartingale with a decomposition*

$$B_t = M_t - \int_0^t \frac{B_u - B_{1-u}}{1-2u}\, du.$$

Here M is a $\{H_t\}$ Brownian motion. Similarly, for $0 \le t < \frac{1}{2}$

$$\overline{B}_t = B_{1-t} = \overline{M}_t + \int_0^t \frac{B_u - B_{1-u}}{1-2u}\, du$$

where $\overline{M}$ is a $\{H_t\}$ Brownian motion independent of M.

Proof. Suppose $0 \le t < \frac{1}{2}$ and $t \le s \le 1-t$. Any Markov process is a Markov field, (see the work of Jamison [6] on reciprocal processes). Therefore, by the Markov field (or reciprocal process) property:

$$E[B_s \mid H_t] = E[B_s \mid B_t,\ B_{1-t}] = E[B_s \mid B_t,\ B_t - B_{1-t}].$$

The random variables are Gaussian and B_t, $B_t - B_{1-t}$ are independent, so this conditional expectation is a projection and equals

$B_t + \dfrac{(t-s)}{(1-2t)}(B_t - B_{1-t})$. Consequently, $E[B_{t+h} - B_t \mid H_t] = \dfrac{-h}{(1-2t)}(B_t - B_{1-t})$. Therefore, $\int_0^{(1/2)-\delta} E\big|E[B_{t+h} - B_t \mid H_t]\big| dt = O(h)$ and so from Theorem 2 of Stricker [8], (see Theorem 3.10 below), B is a $\{H_t\}$ quasimartingale with a unique decomposition:

$$B_t = M_t + \int_0^t a_u du;$$

here M is a $\{H_t\}$ martingale. Now

$$\begin{aligned} E[B_{t+h} - B_t \mid H_t] &= E\Big[\int_t^{t+h} a_u du \mid H_t\Big] \\ &= \frac{-h}{(1-2t)}(B_t - B_{1-t}) \end{aligned}$$

so dividing by $h > 0$ and letting $h \to 0$ we see $a_t = \dfrac{-(B_t - B_{1-t})}{(1-2t)}$. Furthermore, because the quadratic variation of bounded variation terms is zero the quadratic variation process

$$\begin{aligned} \langle B\rangle_t - \langle B\rangle_s &= \lim_{|\Pi|\to 0} \sum_{i=1}^{N} \Big(B_{t_{i+1}} - B_{t_i}\Big)^2 \\ &= \lim_{|\Pi|\to 0} \sum_{i=1}^{N} \Big(M_{t_{i+1}} - M_{t_i}\Big)^2 = \langle M\rangle_t - \langle M\rangle_s \\ &= t - s, \end{aligned}$$

where the limit in probability is taken over partitions $\Pi = \{s \le t_0 < t \le \cdots \le t_N = t\}$ of $[s,t] \subset [0, \frac{1}{2})$ and $|\Pi| = \max_i |t_{i+1} - t_i|$. Therefore $\{M_t\}$ is a continuous $\{H_t\}$ martingale with $\langle M\rangle_t = t$ and so M is a $\{H_t\}$ Brownian motion.

Similarly $\overline{B}_t = B_{1-t} = \overline{M} + \int_0^t \bar{a}_u du$ where $\overline{M}$ is a $\{H_t\}$ mar-

tingale. In fact

$$E[B_{1-t-h} - B_{1-t} \mid H_t] = E[B_{1-t-h} - B_{1-t} \mid B_t,\ B_t - B_{1-t}]$$

$$= B_t + \frac{(2t+h-1)}{(1-2t)}(B_t - B_{1-t}) - B_{1-t}$$

$$= \frac{h}{(1-2t)}(B_t - B_{1-t}) = E\Big[\int_t^{t+h} \bar{a}_u du \mid H_t\Big].$$

As above, $\bar{a}_t = \dfrac{B_t - B_{1-t}}{(1-2t)}$ and $B_{1-t} = \overline{M}_t + \displaystyle\int_0^t \frac{B_u - B_{1-u}}{1-2u}\,du$ where $\overline{M}$ is a $\{H_t\}$ Brownian motion.

The random variables M, $\overline{M}$ are Gaussian, so to establish independence it is sufficient to show they are orthogonal.

$$E[M_t\overline{M}_t] = E\Big[\Big(B_t + \int_0^t \frac{B_u - B_{1-u}}{1-2u}\Big)\Big(B_{1-t} - \int_0^t \frac{B_s - B_{1-s}}{1-2s}\,ds\Big)\Big]$$

$$= t + \int_0^t \frac{t-u}{1-2u}\,du + \int_0^t \frac{t+u-1}{1-2u}\,du$$

$$- 2E\Big[\int_0^t \Big(\int_0^u \frac{B_s - B_{1-s}}{1-2s}\,ds\Big)\frac{B_u - B_{1-u}}{1-2u}\,du\Big]$$

$$= t + (2t-1)\int_0^t \frac{du}{1-2u} - 2\int_0^t \Big(\int_0^u \frac{1-2u}{1-2s}\,ds\Big)\frac{1}{1-2u}\,du$$

$$= t + (2t-1)\int_0^t \frac{du}{1-2u} - 2\int_0^t \Big(\int_0^u \frac{ds}{1-2s}\Big)du$$

$$= t + (2t-1)\int_0^t \frac{du}{1-2u} - 2\int_0^t \frac{(t-s)}{1-2s}\,ds$$

$$= 0.$$

The Brownian motions M and $\overline{M}$ are, therefore, independent.

3. SEMIMARTINGALE DECOMPOSITIONS

For $0 \leq t \leq 1$ consider an n-dimensional Brownian motion $B = (B^1, \dots, B^n)$ defined on a probability space (Ω, F, P). Suppose

the functions a^i, g^{ij} belong to $C^\infty(R^d)$ and satisfy growth conditions of the form

$$\sum_{i=1}^d |a^i(x)|^2 + \sum_{i=1}^d \sum_{j=1}^n |g^{ij}(x)|^2 \le K^2(1+|x|^2).$$

Consider the associated vector fields

$$A(x) = \sum_{i=1}^d a^i(x)\frac{\partial}{\partial x_i}; \quad X_k(x) = \sum_{i=1}^d g^{ik}(x)\frac{\partial}{\partial x_i},$$

for $1 \le k \le n$, and a second order operator

$$L(x) = A(x) + \frac{1}{2}\sum_{i,j=1}^d \Big(\sum_{k=1}^n g^{ik}(x)g^{jk}(x)\Big)\frac{\partial^2}{\partial x_i \partial x_j}. \tag{3.1}$$

Suppose an initial condition x_0 is given which is an R^d-valued random variable independent of $B_t^k - B_s^k$ for $0 \le s \le t \le 1$, $1 \le k \le n$, and independent of the observation process, (see 3.3 below).

Signal 3.1. Consider a signal process which is the solution $\{x_t\}$ of the system

$$dx_t = A(x_t)dt + \sum_{k=1}^n X_k(x_t) \circ dB_t^k. \tag{3.2}$$

Here $\circ dB$ denotes the Stratonovich integral. For any C^3 function $\phi : R^d \to R$

$$\phi(x_t) = \phi(x_0) + \int_0^t A(x_u)\phi(x_u)du + \sum_{k=1}^d \int_0^t X_k(x_u)\phi(x_u) \circ dB_u^k. \tag{3.3}$$

The Ito integral form of (3.3) is

$$\phi(x_t) = \phi(x_0) + \int_0^t L(x_u)\phi(x_u)du + \sum_{k=1}^d \int_0^t X_k(x_u)\phi(x_u)dB_u^k. \tag{3.4}$$

Notation 3.2. For $0 \le t \le 1$ write $\{F_t\}$, resp. $\{\widehat{F}_t\}$, for the right continuous completion of the filtration generated by $\sigma\{x_s : 0 \le s \le t\}$, resp. $\sigma\{x_0,\ B_v - B_u : 0 \le u \le v \le t\}$.

Similarly, $\{G_t\}$, resp. $\{\widehat{G}_t\}$, will denote the right continuous completion of the (reverse time) filtration generated by $\sigma\{x_s : t \le s \le 1\}$, resp. $\sigma\{x_1,\ B_v - B_u : t \le u \le v \le 1\}$.

If $f(u)$, $0 \le u \le 1$, is a $\{G_t\}$ predictable process, continuous in probability and such that $\int_0^1 E[f(u)^2]du < \infty$, the backward Ito integral is defined by Kunita [7] as

$$\int_s^t f(u)\widehat{d}B_u^k = \lim_{|\Pi|\to 0} \sum_{j=0}^{n-1} f(t_{j+1})\Big(B_{t_{j+1}}^k - B_{t_k}^k\Big).$$

Here $\Pi = \{s = t_0 \le t_1 \le \cdots \le t_N = t\}$ is a partition of $[s,t]$ and $|\Pi| = \max_i |t_{j+1} - t_j|$.

As in Elliott and Anderson [3], a reverse time Ito integral form of (3.2) is

$$x_t = x_1 + \int_1^t \widehat{L}(x_u)du + \sum_{k=1}^d \int_1^t X_k(x_u)\widehat{d}B_u^k \tag{3.5}$$

where $\widehat{L}(x) = A(x) - \frac{1}{2}\sum_{i,j=1}^d \Big(\sum_{k=1}^n g^{ik}(x)g^{jk}(x)\Big)\frac{\partial^2}{\partial x_j \partial x_j}$.

Observation 3.3. The signal process is not observed directly but via a noisy observation process $\{y_t\}$ where

$$y_t = \int_0^t h(x_u)du + w_t, \qquad \text{for} \quad 0 \le t \le 1. \tag{3.6}$$

Here $y_t \in R^m$ and $w(w^1,\ldots,w^m)$ is an m-dimensional Brownian motion on (Ω, F, P) which is independent of B and x_0.

Write

$$L^y = \sum_{i=1}^m h^i(x)\,\frac{\partial}{\partial y_i} + \frac{1}{2}\sum_{i=1}^m \frac{\partial^2}{\partial y_i^2}$$

$$\text{and} \quad \widehat{L}^y = \sum_{i=1}^m h^i(x)\frac{\partial}{\partial y_i} - \frac{1}{2}\sum_{i=1}^m \frac{\partial^2}{\partial y_i^2}.$$

Then if $\psi : R^m \to R$ is any C^2 function

$$\psi(y_t) = \psi(0) + \int_0^t L^y\psi(y_u)du + \sum_{i=1}^m \int_0^t \frac{\partial\psi}{\partial y_i}(y_u)dw_u^i. \tag{3.7}$$

Notation 3.4. For $0 \le t \le 1$ write $\{Y_t\}$, resp. $\{\widehat{Y}_t\}$, for the right continuous complete filtration generated by $\sigma\{y_s : 0 \le s \le t\}$, resp. $\sigma\{w_s : 0 \le s \le t\}$, and Z_t, resp. $\widehat{Z}_t$, for the right continuous completion of the filtration generated by $\sigma\{y_s : t \le s \le 1\}$, resp. $\{y_1,\ w_s - w_1 : t \le s \le 1\}$.

Remark 3.5. Although $\{y_t\}$ is not a Markov process, $\{x_t, y_t\}$ is Markov.

We shall require the following hypotheses satisfied:

Hypotheses 3.6. Suppose the diffusion $\{x_t, y_t\}$ is such that:

1. For each $t \in [0,1]$ there is a smooth density $q(t,x,y)$ of (x_t, y_t).
2. For each $s,t \in [0,1]$, $s \le t$, there is a smooth transition density
$$p(x_s, y_s, x_t, y_t, s, t).$$
3. If $\kappa^i(t,x,y) = -\Big(\operatorname{div} X_i(x) + I_{q\neq 0}\dfrac{X_i q(t,x,y)}{q(t,x,y)}\Big)$ for $1 \le i \le n$, and $\lambda^k(t,x,y) = -I_{q\neq 0}\dfrac{\partial q}{\partial y_k}(t,x,y)\cdot q(t,x,y)^{-1}$ for $1 \le k \le m$, then κ^i and $\lambda^k \in L^1([\delta,1]\times\Omega)$ for any $\delta > 0$.
4. κ^i and $\lambda^k \in L^2_{\text{loc}}([\delta,1]\times\Omega)$ for any $\delta > 0$.
5. Consider $p(x,y,\xi,\zeta,s,t)$ and write
$$\ell^k(x,y,\xi,\zeta,s,t) = I_{p\neq 0}\frac{\partial p}{\partial y_k}\cdot p^{-1}$$
$$\bar{\ell}_k(x,y,\xi,\zeta,s,t) = I_{p\neq 0}\frac{\partial p}{\partial \zeta_k}\cdot p^{-1}.$$
Then $\ell^k(x_t,\ y_t,\ x_{1-t},\ y_{1-t},\ t,\ 1-t) \in L^1([0,\frac{1}{2}-\delta]\times\Omega)$ and $\bar{\ell}^k(x_t,\ y_t,\ x_{1-t},\ y_{1-t},\ t,\ 1-t) \in L^1([0,\frac{1}{2}-\delta]\times\Omega)$ for any $\delta \in (0,\frac{1}{2})$, $1 \le k \le m$.

From Elliott and Anderson [3] we can quote:

Theorem 3.7. *Under Hypotheses 3.6(1), (3) and (4)*

$$\{B_t - B_1\} = \{B_t^1 - B_1^1, \dots, B_t^n - B_1^n\}$$

and

$$\{w_t - w_1\} = \{w_t^1 - w_1^1, \dots, w_t^m - w_1^m\}$$

are reverse time $\{\widehat{G}_t \vee \widehat{Z}_t\}$ *quasimartingales. In fact, if* $\widetilde{B}^i_t = B^i_t - B^i_1 - \int_t^1 \kappa^i(u, x_u, y_u)du$ *for* $1 \leq i \leq n$, *and* $\widetilde{w}^k_t = w^k_t - w^k_1 - \int_t^1 \lambda^k(u, x_u, y_u)du$ *for* $1 \leq k \leq m$, *then* $\{\widetilde{B}, \widetilde{w}\}$ *is an* $n+m$ *dimensional* $\{\widehat{G}_t \vee \widehat{Z}_t\}$ *Brownian motion.*

Notation 3.8. Consider now the two-sided σ-fields for $0 \leq t \leq \frac{1}{2}$:

$$H_t = F_t \vee G_{1-t} \quad \text{and} \quad \widehat{H}_t = \widehat{F}_t \vee \widehat{G}_{1-t} \quad \text{for the signal } x,$$

$$K_t = Y_t \vee Z_{1-t} \quad \text{and} \quad \widehat{K}_t = \widehat{Y}_t \vee \widehat{Z}_{1-t} \quad \text{for the observation } y, \text{ and}$$

$$H_t \vee K_t, \quad \widehat{H}_t \vee \widehat{K}_t \quad \text{for } (x, y).$$

In a manner similar to [1] and Section 2 we shall now determine the semimartingale decompositions of $\{w_t\}$ and $\{\bar{w}_t\} = \{w_{1-t} - w_1\}$ with respect to the filtration $\{\widehat{H}_t \vee \widehat{K}_t\}$.

Theorem 3.9. *Suppose Hypotheses 3.6 are satisfied. Then for* $1 \leq k \leq m$

$$\lim_{h \to 0+} h^{-1} E[w^k_{t+h} - w^k_t \mid \widehat{H}_t \vee \widehat{K}_t] = \ell^k(x_t,\ y_t,\ x_{1-t},\ y_{1-t},\ t,\ 1-t) \tag{3.8}$$

weakly in $L^1(\Omega)$.

If $\bar{w}_t = w_{1-t} - w_1$,

$$\lim_{h \to 0+} h^{-1} E[\bar{w}^k_{t+h} - \bar{w}^k_t \mid \widehat{H}_t \vee \widehat{K}_t]$$

$$= \lim_{h \to 0+} h^{-1} E[w^k_{1-t-h} - w^k_{1-t} \mid \widehat{H}_t \vee \widehat{K}_t]$$

$$= \bar{\ell}^k(x_t,\ y_t,\ x_{1-t},\ y_{1-t},\ t,\ 1-t) \tag{3.9}$$

weakly in $L^1(\Omega)$.

Note the right hand sides of (3.8) and (3.9) are $H_t \vee K_t$ measurable.

Proof. The proof uses the reverse time differentiation rule and Stricker's characterization of quasimartingales ([8], Theorem 2). Details can be found in [4].

We then have

Theorem 3.10. *Write*

$$\ell = (\ell^1, \ldots, \ell^m)$$

$$\bar{\ell} = (\bar{\ell}^1, \ldots, \bar{\ell}^m).$$

Then $\{w_t\}$ and $\{\bar{w}_t\}$ are $\{\widehat{H}_t \vee \widehat{K}_t\}$ quasimartingales with decompositions

$$w_t = \beta_t + \int_0^t \ell(x_u,\ y_u,\ x_{1-u},\ y_{1-u},\ u,\ 1-u)du$$

$$\bar{w}_t = w_{1-t} = \bar{\beta}_t + \int_0^t \bar{\ell}(x_u,\ y_u,\ x_{1-u},\ y_{1-u},\ u,\ 1-u)du.$$

Here β and $\bar{\beta}$ are independent m-dimensional $\{\widehat{H}_t \vee \widehat{K}_t\}$ Brownian motions.

4. BILATERAL FILTERING

Notation 4.1. Π will denote the predictable projection with respect to the two-sided complete, right continuous filtration $\{K_t\}$ generated by y_s, $0 \le s \le t$, and $1-t \le s \le 1$.

The 'forward' part of the observation process is

$$y_t = \int_0^t h(x_u)du + w_t. \tag{4.1}$$

With respect to the filtration $\{\widehat{H}_t \vee \widehat{K}_t\}$ this can be written

$$y_t = \int_0^t h(x_u)du + \int_0^t \ell(u)du + \beta_t \tag{4.2}$$

where β is an $\{\widehat{H}_t \vee \widehat{K}_t\}$ Brownian motion. Taking the $\{K_t\}$ projections this can be expressed as

$$y_t = \int_0^t \Pi h(x_u)du + \int_0^t \Pi\ell(u)du + \nu_t. \tag{4.3}$$

Here ν_t is $\{K_t\}$ adapted and

$$\nu_t = \int_0^t (h(x_u) - \Pi h(x_u))du + \int_0^t (\ell(u) - \Pi\ell(u))du + \beta_t.$$

Therefore, ν is a continuous $\{K_t\}$ martingale. As in [2] the product rule show that

$$\langle \nu^i,\ \nu^j \rangle_t = \langle \beta^i,\ \beta^j \rangle_t = \delta_{ij} t$$

for $1 \leq i, j \leq m$, so ν is a $\{K_t\}$ Brownian motion. Now

$$y_1 = \int_0^1 h(x_u)du + w_1.$$

Consequently the 'reverse time' part of the observation process can be written

$$\begin{aligned}
\bar{y}_t = y_{1-t} &= y_1 - \int_{1-t}^1 h(x_u)du + w_{1-t} - w_1 \\
&= y_1 - \int_{1-t}^1 h(x_u)du + \int_0^t \bar{\ell}(u)du + \bar{\beta}_t \\
&= \bar{y}_0 - \int_0^t h(x_{1-u})du + \int_0^t \bar{\ell}(u)du + \bar{\beta}_t. \qquad (4.4)
\end{aligned}$$

Taking the $\{K_t\}$ projection this can be written

$$\bar{y}_t = \bar{y}_0 - \int_0^t \Pi h(x_{1-u})du + \int_0^t \Pi\bar{\ell}(u)du + \bar{\nu}_t \qquad (4.5)$$

where, as above, $\bar{\nu}$ is a $\{K_t\}$ Brownian motion independent of ν

We can now derive the bilateral prediction formula:

Theorem 4.2. *Consider the signal and observation processes determined by (3.2) and (3.6), respectively. Suppose F is any real valued C^2 function with compact support defined on R^d. For $0 \leq s \leq \frac{1}{2}$ write*

$$\Lambda_s = E\big[F(x_{1/2}) \mid H_s \vee K_s\big].$$

Then

$$\begin{aligned}
\Pi(\Lambda_t) &= E[F(x_{1/2}) \mid K_t] \\
&= \Pi(\Lambda_0) + \int_0^t \Big\{ \Pi\big(\Lambda_u(h(x_u) + \ell(u))\big) \\
&\qquad - \Pi(\Lambda_u)\big(\Pi(h(x_u)) + \Pi(\ell(u))\big) \Big\} d\nu_u \\
&\quad + \int_0^t \Big\{ \Pi\big(\Lambda_u(\bar{\ell}(u) - h(x_{1-u}))\big) \\
&\qquad + \Pi(\Lambda_u)\big(\Pi(h(x_{1-u})) - \Pi(\bar{\ell}(u))\big) \Big\} d\bar{\nu}_u.
\end{aligned}$$

Proof. First note that Λ is introduced for notational convenience and because, for example,

$$\Pi(\Lambda_t) = E[F(x_{1/2}) \mid K_t]$$

$$\Pi(\Lambda_u(\bar{\ell}(u))) = E[E[F(x_{1/2}) \mid H_u \vee K_u]\bar{\ell}_u \mid K_u]$$
$$= E[F(x_{1/2})\bar{\ell}(u) \mid K_u]$$

the final equation could be written just in terms of $F(x_{1/2})$.

$$\Lambda_t = E[F(x_{1/2}) \mid H_t \vee K_t] \tag{4.6}$$

is a martingale by definition and $\Pi(\Lambda_t) = E[F(x_{1/2}) \mid K_t]$ is a $\{K_t\}$ martingale. Now Λ is the solution of a prediction or smoothing problem, and as in Theorem 16.22 of [2], Λ_t has a representation as a stochastic integral.

$$\Lambda_t = \Lambda_0 + \int_0^t \alpha_u dB_u + \int_t^1 \bar{\alpha}_u dB_u.$$

The nature of the integrands α, $\bar{\alpha}$ could be investigated. However, this would not contribute to the solution, because what is required is a recursive expression for $\Pi(\Lambda_t)$. Now again, $\Pi(\Lambda_t)$ has a representation as a stochastic integral.

$$\Pi(\Lambda_t) = \Pi(\Lambda_0) + \int_0^t \gamma_u d\nu_u + \int_0^t \bar{\gamma}_t\, d\bar{\nu}_u. \tag{4.7}$$

We wish to determine the processes γ and $\bar{\gamma}$. Forming the products of Λ with (4.2) and (4.4)

$$\Lambda_t y_t = \int_0^t \Lambda_u h(x_u)du + \int_0^t \Lambda_u \ell(u)du$$
$$+ \int_0^t \Lambda_u d\beta_u + \int_0^t y_u d\Lambda_u + \langle \Lambda, y\rangle_t \tag{4.8}$$

$$\Lambda_t \bar{y}_t = \Lambda_0 \bar{y}_0 - \int_0^t \Lambda_u h(x_{1-u})du + \int_0^t \Lambda_u \bar{\ell}(u)du$$
$$+ \int_0^t \Lambda_u d\bar{\beta}_u + \int_0^t \bar{y}_u d\Lambda_u + \langle \Lambda, \bar{y}\rangle_t. \tag{4.9}$$

However, because $\langle B, y\rangle = 0$, $\langle B, \bar{y}\rangle = 0$, the quadratic variation terms in (4.8) and (4.9) vanish. Taking the $\{K_t\}$ projection of both sides of (4.8)

$$\Pi(\Lambda_t y_t) = \Pi(\Lambda_t) y_t$$

$$= \int_0^t \Pi(\Lambda_u(h(x_u) + \ell(u))du + M_t \tag{4.10}$$

where M is a $\{K_t\}$-martingale. Similarly, taking the $\{K_t\}$ projection of both sides of (4.9):

$$\Pi(\Lambda_t \bar{y}_t) = \Pi(\Lambda_t)\bar{y} = \Pi(\Lambda_0)\bar{y}_0$$

$$+ \int_0^t \Pi(\Lambda_u(\bar{\ell}(u) - h(x_{1-u})))du + \overline{M}_t, \tag{4.11}$$

where $\overline{M}$ is a $\{K_t\}$-martingale. However, if we take the product of $\Pi(\Lambda_t)$, as given by (4.7), and (4.3)

$$\Pi(\Lambda_t)y_t = \int_0^t \Pi(\Lambda_u)\big(\Pi h(x_u) + \Pi \ell(u)\big)du$$

$$+ \int_0^t y_u \gamma_u d\nu_u + \int_0^t y_u \bar{\gamma}_u d\bar{\nu}_u + \int_0^t \gamma_u du + N_t \tag{4.12}$$

where N is a $\{K_t\}$ martingale. The stochastic integrals with respect to ν and $\bar{\nu}$ are also $\{K_t\}$ martingales. The process $\Pi(\Lambda_t)y_t$ is clearly a special semimartingale, so the decompositions (4.10) and (4.12) must be the same.

Equating the bounded variation terms we have

$$\gamma_t = \Pi(\Lambda_t(h(x_t) + \ell(t))) - \Pi(\Lambda_t)(\Pi h(x_t) + \Pi \ell(t)). \tag{4.13}$$

Similarly, forming the product of (4.7) and (4.5):

$$\Pi(\Lambda_t)\bar{y}_t = \Pi(\Lambda_0)\bar{y}_0 + \int_0^t \Pi(\Lambda_u)(\Pi\bar{\ell}(u) - \Pi h(x_{1-u}))du$$

$$+ \int_0^t \bar{y}_u \gamma_u d\nu_u + \int_0^t \bar{y}_u \bar{\gamma}_u d\bar{\nu}_u + \int_0^t \bar{\gamma}_u du + \overline{N}_t. \tag{4.14}$$

Again, the decompositions (4.11) and (4.14) must be the same, so equating their bounded variation terms we see

$$\tilde{\gamma}_t = \Pi\big(\Lambda_t(\bar{\ell}(t) - h(x_{1-t}))\big) + \Pi(\Lambda_t)\big(\Pi h(x_{1-t}) - \Pi\bar{\ell}(t)\big). \quad (4.15)$$

Substituting (4.14) and (4.15) into (4.7) the result follows.

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